Area Quantization in Quasi-Extreme Black Holes

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Abstract

We consider quasi-extreme Kerr and quasi-extreme Schwarzschild-de Sitter black holes. From the known analytical expressions obtained for their quasi-normal modes frequencies, we suggest an area quantization prescription for those objects.

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The question of the quantization of the black hole horizon area is well posed and has been considered long ago by Bekenstein [1], being a major issue since then [2], [3], [4], [5], [6], [7], [8], [9]. The microscopic origin of the black hole entropy [10], [11] is also an unanswered question. There are attempts to partially understand these questions using string theory [12], [13] as well as the canonical approach of quantum gravity [14], [15], [16], [17]. Recently, the quantization of the black hole area has been considered [5], [6] as a result of the absorption of a quasi-normal mode excitation. Bekenstein's idea for quantizing a black hole is based on the fact that its horizon area, in the nonextreme case, behaves as a classical adiabatic invariant [1], [4]. It is worthwhile studying how quasi-extreme holes would be quantized. It is specially interesting to investigate this case since we analytically know the quasi-normal mode spectrum of some black holes of that kind, namely the quasi-extreme Kerr [18] and quasi-extreme Schwarzschild-de Sitter (*i.e.*, near-Nariai) [19] solutions. The quasi-normal modes of black holes are the characteristic, ringing frequencies which result from their perturbations [20] and provide a unique signature of these objects [21], possible to be observed in gravitational waves. Besides, quasi-normal modes have been used to obtain further information of the space-time structure, as for example in [22], [23], [24], [25], and [26].

Furthermore, gravity in such extreme configurations are an excellent laboratory for the understanding of quantum gravity, and information about the quantum structure of space-time can be derived in such contexts by means of general setups [27].

The first case of interest to us where the black hole quasi-normal mode spectrum is analytically known is the quasi-extreme Kerr black hole. In this case, the specific angular momentum of the hole, a, is very nearly its mass M ($a \approx M$). Detweiler [28] was able to show that in such a case there is an infinity of quasi-normal modes given by [18]

$$\omega_n M \approx \frac{m}{2} - \frac{1}{4m} \exp[\frac{\xi - 2n\pi}{2\delta} + i\eta] \,, \tag{1}$$

where $n = 0, 1, ...$ labels the solution, m is an integer labeling the axial mode of the perturbation, while ξ, δ , and η are constants. We note that (1) is valid for $\ell = m$, where ℓ is the multipole index of the perturbation. For details we refer the reader to [18].

In Boyer-Lindqüist coordinates the Kerr solution reads

$$ds^2 = -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \\ + \left(r^2 + a^2 + 2Ma^2 r\sin^2\theta\right)\sin^2\theta d\phi^2, \quad (2)$$

where

$$\Delta = r^2 - 2Mr + a^2\,, \quad (3)$$

$$\Sigma = r^2 + a^2\cos^2\theta\,. \quad (4)$$

M and $0 \leq a \leq M$ are the black hole mass and specific angular momentum ($a = J/M$), respectively. The horizons are at $r_\pm = M \pm \sqrt{M^2 - a^2}$.

In units $G = c = 1$, the black hole horizon area and its surface gravity (temperature) are given, respectively, by

$$A = 4\pi(r_+^2 + a^2)\,. \quad (5)$$

$$\kappa = \frac{1}{4A}(r_+ - r_-)\,. \quad (6)$$

Based on *Bohr's correspondence principle* ("for large quantum numbers, transition frequencies should equal classical frequencies"), Hod [5] has considered the asymptotic limit $n \to \infty$ for the quasi-normal mode frequencies ω_n of a Schwarzschild black hole in order to determine the spacing of its equally spaced quantum area spectrum. That asymptotic quasi-normal mode spectrum was obtained numerically by Nollert [29]. Recently, Motl [8] has computed analytically $Re(\omega_n)$ as $n \to \infty$, finding agreement with the numerical value of Nollert. In this large n limit, Hod [5] then assumed that a Schwarzschild black hole mass should increase by $\delta M = \hbar Re(\omega_n)$ when it absorbs a quantum of energy $\hbar Re(\omega_n)$.

As in Ref. [5], we expect that the real part of the quasi-normal mode frequency for large n corresponds to an addition of energy equal to $\hbar Re(\omega_n)$ to the quasi-extreme Kerr black hole mass as it falls into its event horizon. Then, taking the limit $n \to \infty$ for ω_n in (1) we simply have

$$\omega_n \approx \frac{m}{2M} \quad , \quad n \to \infty\,. \quad (7)$$

Contrary to the Schwarzschild case, where the limit $n \to \infty$ gives highly damped modes, for the present case, it gives virtually *undamped* modes with frequencies close to the upper limit of the superradiance interval [30], $0 < \omega <$

$m\Omega$, where $\Omega = 4\pi a/A$ is angular velocity of the horizon. The quasi-normal mode spectrum (1) of near-extreme Kerr black holes leads to interesting consequences, as recently analysed by Glampedakis and Anderson [18].

Furthermore, here the angular momentum $\hbar m$ adds to the angular momentum $J = Ma$ associated with the Kerr solution. We then have a pair of variations for black hole parameters given by

$$\delta M = \frac{\hbar m}{2M} \quad ; \qquad \delta J = \hbar m \qquad \Rightarrow \qquad \delta a = \hbar m(\frac{1}{M} - \frac{a}{2M^2})\,. \tag{8}$$

In what follows we will consider $\hbar = 1$ and for the sake of brevity $m = 1$.

The variation of the horizon area is related to the first law of black hole thermodynamics,

$$\delta M = \kappa \delta A + \Omega \delta J\,. \tag{9}$$

Making use of relations (8), we can obtain from (5) that, for a near-extreme hole ($a \approx M$), the area variation is given by

$$\delta A = 8\pi \left(1 + \sqrt{\frac{M-a}{2M}}\right) \quad , \tag{10}$$

up to first order in $(M-a)^{1/2}$.

Therefore, for strictly extreme holes, we simply have $\delta A = 8\pi$.

For $a \approx M$, we can express κ and Ω, respectively, as

$$\kappa \approx \frac{1}{16\pi}\frac{\sqrt{M^2-a^2}}{M^2}\left[1 - \frac{\sqrt{M^2-a^2}}{M}\right] \tag{11}$$

and

$$\Omega \approx \frac{a}{2M^2}\left(1 - \frac{\sqrt{M^2-a^2}}{M} + \frac{M^2-a^2}{M^2}\right)\,. \tag{12}$$

Finally, from (8), (10), (11), and (12), to order $(M-a)^{3/2}$, we obtain

$$\kappa \delta A + \Omega \delta J \approx \frac{1}{2M}\,, \tag{13}$$

in agreement with the first law of black hole thermodynamics (9). Thus we can prescribe the quantization of a quasi-extreme Kerr black hole area as

$$A_n = n\delta A \ell_P^2 \simeq 8\pi \ell_P^2\, n\,, \tag{14}$$

where $n = 1, 2, ...$ and ℓ_P is the Planck length.

A second case where we can obtain information about the black hole parameters involved in its quantization is the near-extreme Schwarzschild-de Sitter (S-dS) black hole. This is the case when the mass of the black hole is increased as to arrive near the limit $M_N = R/3\sqrt{3}$, where the constant R is related to the cosmological constant Λ by $R^2 = 3/\Lambda$. This is the Nariai limit [31], for which the black hole and cosmological horizons coincide. The S-dS metric is [32]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{15}$$

where

$$f(r) = 1 - \frac{2M}{r} - \frac{r^2}{R^2}\,, \tag{16}$$

and $0 \leq M \leq M_N$ is the black hole mass. The roots of $f(r)$ are r_b, r_c and $r_0 = -(r_b + r_c)$, where r_b and r_c are the black hole and cosmological horizon radii, respectively. To each horizon there is a surface gravity, given by $\kappa_{b,c} = \frac{1}{2}\frac{df}{dr}|_{r=r_{b,c}}$. For κ_b we have the expression

$$\kappa_b = \frac{(r_c - r_b)(r_b - r_0)}{2R^2 r_b}\,. \tag{17}$$

As in the Kerr case, we will perform the variation of M. It is useful to write M and R in terms of r_b and r_c as

$$2MR^2 = r_b r_c(r_b + r_c)\,, \tag{18}$$

$$R^2 = r_b^2 + r_b r_c + r_c^2\,. \tag{19}$$

The analytical quasi-normal mode spectrum for the quasi-extreme S-dS black hole has been recently derived by Cardoso and Lemos [19] and reads

$$\omega_n = \kappa_b \left[\sqrt{\frac{V_0}{\kappa_b^2} - \frac{1}{4}} - i(n + \frac{1}{2})\right]\,, \tag{20}$$

where $n = 0, 1, \ldots,$ and $V_0 = \kappa_b^2 \ell(\ell + 1)$, for scalar and electromagnetic perturbations, and $V_0 = \kappa_b^2(\ell + 2)(\ell - 1)$ for gravitational perturbations.

Since we are considering the near extreme limit of the S-dS solution, for which $(r_c - r_b)/r_b << 1$, it is suitable for our purposes to write the black hole mass as

$$M = M_N + \mu = \frac{R}{3\sqrt{3}} + \mu\,. \tag{21}$$

Therefore, since $R=\sqrt{3/\Lambda}$ is fixed, the use of (18) and (19) leads us to

$$\delta M = \delta\mu = \frac{r_b \Delta r \delta r_b}{2R^2}\,, \tag{22}$$

where $\Delta r = r_c - r_b$.

Similarly as we did for the Kerr case, here we can consider $\delta M = \hbar Re(\omega_n)$ and in view of (20) and (17) write ($\hbar = 1$)

$$\delta M = \frac{\Delta r}{2{r_b}^2}\sqrt{(\ell+2)(\ell-1)-\frac{1}{4}}\quad , \tag{23}$$

where we have used $R^2 \sim 3r_b^2$ and considered V_0 for gravitational perturbations.

The variation of the black hole horizon area,

$$\delta A_b = 8\pi r_b \delta r_b\,, \tag{24}$$

then gives us

$$\delta A_b = 24\pi\sqrt{(\ell+2)(\ell-1)-\frac{1}{4}}\quad , \tag{25}$$

for gravitational quasi-normal modes and $\delta A_b = 24\pi\sqrt{\ell(\ell+1)-\frac{1}{4}}$ otherwise.

Thus we can prescribe the quantum area spectrum for a quasi-Nariai black hole as

$$A_{b_n} = n\delta A_b \ell_P^2 \simeq 12\pi\sqrt{15}\ell_P^2\, n\,, \tag{26}$$

where $n = 1, 2, \dots$, or, in the case of scalar or electromagnetic perturbations, $12\pi\sqrt{7}\ell_P^2\, n$.

In summary, with the knowledge of the analytical quasi-normal mode spectrum of near extreme Kerr and near extreme S-dS black holes, as given in [18] and [19], we have prescribed how their horizon area would be quantized. This was done by simply assuming they have a uniformly spaced area spectrum given by $A_n = \delta A \ell_P^2\, n$, where δA is the area variation caused by absorption of a quasi-normal mode. This was done in analogy with the Schwarzschild case, where the spacing of its area spectrum was determined by means of the knowledge of its asymptotic ("large n ") quasi-normal mode frequencies [5]. In the cases regarded here, the results for the spacing of the area spectrum differ from that for Schwarzschild, as well as for non-extreme Kerr [33] black holes, in which cases, the spacing is predicted to be given by

$4 ln\, 3$. This factor comes from the real part of the asymptotic quasi-normal mode frequencies of those black holes [5], [33]. Such a difference may be justified due to the quite different nature of the asymptotic quasi-normal mode spectrum of the near extreme black holes we considered. Furthermore, it should be no *a priori* reason for expecting the same behaviour for the asymptotic quasi-normal mode frequencies of near extreme and non-extreme black holes.

Acknowledgements

The authors would like to thank **CNPq-Brazil** (*Conselho Nacional de Desenvolvimento Científico e Tecnológico*) and **FAPESP-Brazil** (*Fundação de Amparo à Pesquisa no Estado de São Paulo*) for financial support.

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